RESTRICTING SEMISTABLE BUNDLES ON THE PROJECTIVE PLANE TO CONICS

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Abstract. We study the restrictions of rank 2 semistable vector bundles E on $\mathbb{P}^2$ to conics. A Grauert-Mülich type theorem on the generic splitting is proven. The jumping conics are shown to have the scheme structure of a hypersurface $J_2 \subset \mathbb{P}^5$ of degree $c_2(E)$ when $c_1(E) = 0$ and of degree $c_2(E)-1$ when $c_1(E) = -1$. Some examples of jumping conics and jumping lines are studied in detail.

1. Introduction

A standard method in the theory of vector bundles on projective spaces is to restrict a bundle to a line where it splits into a sum of line bundles and then study how the splitting changes as the line varies. Grothendieck's result ([6],[16, page 22]) gives the splitting of a rank r bundle E on $\mathbb{P}^n$ restricted to a line $L \subset \mathbb{P}^n$ as $E_L \cong \bigoplus_{i=1}^r \mathcal{O}_L(a_i)$ for integers $a_1 \geq a_2 \cdots \geq a_r$. The minimal (lexicographic order) splitting occurs for a Zariski open subset (of the Grassmannian $\mathcal{G}$) of lines in $\mathbb{P}^n$ and such lines are called generic lines for E. The lines on which E has a larger splitting are called jumping lines and form a proper closed subscheme J_1 of $\mathcal{G}$. When E is semistable, the Grauert-Mulich theorem states that the generic splitting satisfies $a_j - a_{j+1} \leq 1 \ \forall j$ ([16, page 192] and [2, in the $r = 2$ case]). Further restricting to rank 2 and normalizing E so that $c_1(E) = 0$ or -1, we have that the generic splitting is $\mathcal{O}_L \oplus \mathcal{O}_L$ when $c_1(E) = 0$ and $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$ when $c_1(E) = -1$. In the $c_1(E) = 0$ case, Barth proved [2] that the jumping lines form a hypersurface in $\mathcal{G}$ of degree $c_2(E)$. When $c_1(E) = -1$ and $n = 2$, Hulek showed [10] that there are usually only $\binom{c_2(E)}{2}$ jumping lines but that the jumping lines of the second kind form a curve $\tilde{J}_1 \subset \mathbb{P}^{2^*}$ of degree $2(c_2(E)-1)$. A jumping line of the second kind is a line L such that, for $L^{(1)}$ the first order neighborhood of L in $\mathbb{P}^2$, $E_{L^{(1)}}$ has a non-trivial global section.

In this paper we analyze the restriction of rank 2 semistable bundles E on $\mathbb{P}^2$ to conics. One complication arises from the fact that there are three kinds of conics: smooth conics C isomorphic to $\mathbb{P}^1$ via a quadratic parameterization $\mathbb{P}^1 \xrightarrow{g} C$, unions of distinct lines $L_1 + L_2$, and double lines $2L$ (=first order neighborhood $L^{(1)}$ of L in $\mathbb{P}^2$). For smooth conics we prove a Grauert-Mulich type theorem for semistable E of arbitrary rank (Theorem 1). When the rank of E is 2,

$$c_1(g^*E) = \begin{cases} 0 & \text{if } c_1(E) = 0 \\ -2 & \text{if } c_1(E) = -1 \end{cases}$$

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and so

$$E_C \cong g^*E \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a) & \text{for } c_1(E) = 0 \\ \mathcal{O}_{\mathbb{P}^1}(-1+a) \oplus \mathcal{O}_{\mathbb{P}^1}(-1-a) & \text{for } c_1(E) = -1 \end{cases} \tag{1.1}$$

with $a = 0$ corresponding to a generic smooth conic and $a > 0$ defining a smooth jumping conic. For the reduced case, jumping conics are defined as follows. When $c_1(E) = 0$, $C = L_1 + L_2$ is a jumping conic means that L_1 or L_2 is a jumping line. When $c_1(E) = -1$, $C = L_1 + L_2$ (L_1 and L_2 distinct) is a jumping conic if either L_1 or L_2 is jumping or if both have generic splitting $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and their $\mathcal{O}_{\mathbb{P}^1}$ factors match up at the intersection point $L_1 \cap L_2$. When $c_1(E) = -1$, and $C = 2L$, a jumping conic is a jumping line of the second kind. We prove that these definitions fit together by showing that the jumping conics have the subscheme structure of a hypersurface J_2 in $\mathbb{P}^5$, of degree $c_2(E)$ if $c_1(E) = 0$ and of degree $c_2(E) - 1$ if $c_1(E) = -1$(Theorem 2). Furthermore, the singular jumping conics are in the scheme-theoretic closure of the smooth jumping conics.

The proof of this theorem, easy in the $c_1(E) = -1$ case, requires two substantial ingredients when $c_1(E) = 0$: 1.the theory of stable bundles on ruled surfaces and 2.Hurtubise's analysis of families of rank 2 bundles on $\mathbb{P}^1$. The proof also gives a new demonstration and explanation of another result of Barth. Denote the intersection index of a line Λ in $\mathbb{P}^5$ and J_2 at $C \in J_2$ by $(\Lambda \cdot J_2)_C$. Define the jump size of C by the integer a in (1.1). Then $(\Lambda \cdot J_2)_C \geq a$ (Corollary 2). Actually, Barth's result is the corresponding statement for jumping lines [2, Section 6].

Theorem 2 also relates to Hulek's result on the curve $\tilde{J}_1$ of jumping lines of the second kind. The double lines comprise the degree 2 image Δ of $\mathbf{P}^{2*}$ in the space of conics $\mathbb{P}^5$ via the Veronese map. Since $\tilde{J}_1 = \Delta \cap J_2$ and since deg $J_2 = c_2(E) - 1$, deg $\tilde{J}_1 = 2(c_2(E) - 1)$. Of course this depends on the fact that not all lines are jumping lines of the second kind and this is the major part of Hulek's proof.

In Section 5, Section 6, and Section 7, we describe some examples of jumping conics and jumping lines on rank 2 stable bundles E on $\mathbb{P}^2$ in some detail. When $c_1 = 0$ and $c_2 = 2$, E is determined by a map into the grassmannian of lines in $\mathbb{P}^3$ and and the jumping lines and conics are described relative to this map and the Schubert cycles in the grassmannian. The $c_1 = 0$, $c_2 = 3$ bundles are of two types. The generic bundle is determined by a map $f \colon \mathbb{P}^2 \to \mathbb{P}^2$ and the jumping lines and conics can be studied in terms of this map and its ramification divisor. A non-generic bundle is obtained from $T\mathbb{P}^2$ by an elementary modification along a unique line and its jumping lines and conics can be understood via this description. Every $c_1 = -1$, $c_2 = 2$ bundle is an elementary modification of the trivial bundle along a line. From this we obtain an explicit equation for its hyperplane of jumping conics.

Ran has obtained [17] a Grauert-Mulich type theorem and results on jumping curves for rank r semistable bundles on $\mathbb{P}^n$ and their restrictions to rational curves under certain conditions. His methods are related to quantum K-theory.

After this paper was written, we learned of the earlier paper of Manaresi [13] on the same subject. Our definition of jumping conics is the same as her definition as is our result on the degree of the hypersurface of jumping conics. The methods of our paper are quite different from those of [13]. In the $c_1 = 0$ case, she constructs a complete simultaneous deformation of a rank two bundle on two lines meeting at

a point, trivial on one line and non-trivial on the other, to a trivial bundle on a smooth conic. This is used to show that the jumping conics form a hypersurface. We do not understand how her results show that the degree of the hypersurface is c_2.

2. Preliminaries

By a stable bundle we shall mean Mumford-stable (or μ- stable), that is

Definition 1. Let X be a smooth projective variety of dimension n, $\mathcal{O}_X(1)$ a very ample line bundle on X, and H a corresponding hyperplane section of X. A coherent torsion-free rank r sheaf E on X is called stable (resp.:. semistable) if, for any subsheaf $F \subset E$ of rank $r_1 < r$, $c_1(F) \cdot H^{n-1}/r_1 < c_1(E) \cdot H^{n-1}/r$ (resp.:. $\leq$).

When E is a rank 2 bundle on $\mathbb{P}^2$ normalized so that $c_1(E) = 0$ or -1, stability is equivalent to $h^0(\mathbb{P}^2; E) = 0$ when $c_1(E) = 0$ and to $h^0(\mathbb{P}^2; E(-1)) = 0$ when $c_1(E) = -1$. For $c_1(E) = 0$, semistability is equivalent to $h^0(\mathbb{P}^2; E(-1)) = 0$ [16, Ch.2, Sec.1.2]. The stable bundles on $\mathbb{P}^2$ of fixed Chern classes are parameterized by a coarse moduli scheme $\mathcal{M}(c_1, c_2)$ which is a quasi-projective variety [15]. By deformation theory, $\mathcal{M}(c_1, c_2)$ is smooth and $\dim \mathcal{M}(0, c_2) = 4c_2 - 3$, $\dim \mathcal{M}(-1, c_2) = 4c_2 - 4$.

The Riemann-Roch formula for a rank r bundle on $\mathbb{P}^2$ [7, Append.A,Sec.4] is

$$\chi(\mathbb{P}^2; E(k)) = \begin{cases} \frac{r}{2}(k+2)(k+1) - c_2(E) & \text{for } c_1(E) = 0 \\ \frac{r}{2}(k+2)(k+1) - k - 1 - c_2(E) & \text{for } c_1(E) = -1. \end{cases} \tag{2.1}$$

For a rank r bundle on a smooth genus g curve Y, Riemann-Roch is

$$\chi(Y; E(k)) = r(1-g) + rk + c_1(E). \tag{2.2}$$

For E a rank 2 bundle on a compact surface X, $D \overset{j}{\hookrightarrow} X$ an effective divisor, $\mathcal{L}$ a line bundle on D, and $\psi \colon E \to E|_D \to \mathcal{L}$ a bundle surjection, define the rank two bundle E' by the elementary modification

$$0 \longrightarrow E' \longrightarrow E \xrightarrow{\psi} j_*\mathcal{L} \longrightarrow 0. \tag{2.3}$$

Then [4, page 41]

$$c_1(E') = c_1(E) - [D]^* \quad \in H^2(X; \mathbf{Z}) \tag{2.4}$$

$$c_2(E') = c_2(E) - [D]^* \cdot c_1(E) + c_1(\mathcal{L}) \quad \in \mathbf{Z} \tag{2.5}$$

where $[D]^*$ denotes the cohomology class dual to the homology class $[D]$ determined by the divisor D.

3. Generic Splitting on Conics

A conic C_ξ in $\mathbb{P}^2 = \mathbf{P}(V)$ is defined by the vanishing of a non-trivial homogeneous polynomial of degree two, $\xi = \sum_{i,j=0}^{2} \xi_{ij} x_i x_j = 0$. We let ξ denote this polynomial or the symmetric 3 by 3 matrix $\xi = (\xi_{ij})$ or the corresponding element of $S^2(V^*)$ as needed. We also use ξ to represent the corresponding projective class in $\mathbb{P}^5 \equiv$

$\mathbf{P}(S^2(V^*))$. C_ξ is a smooth conic if and only if $\det(\xi) \neq 0$ and so the singular conics form a degree three hypersurface $S \subset \mathbb{P}^5$. For $\xi \in S$, $C_\xi = L + L^{'}$ where L is a line defined by $l(x) = \sum_{i=0}^{2} l_i x_i = 0$ with a similar equation for L'. Then $\mathbb{P}^{2^*} \times \mathbb{P}^{2^*} \to S$ defined by $(l, l') \mapsto l \cdot l'$ induces an isomorphism of the second symmetric power of $\mathbb{P}^{2^*}$ and S. The diagonal Δ corresponds to the double lines $C_\xi = 2L$.

A conic in $\mathbb{P}^n$ for $n \geq 3$ lies in a 2-plane $\mathbb{P}^2 \subset \mathbb{P}^n$. If $\mathcal{G}_{2,n}$ is the grassmannian of linear $\mathbb{P}^2$'s in $\mathbb{P}^n$ and $\mathcal{S}$ is the tautological sub-bundle (rank 3) on $\mathcal{G}_{2,n}$, the conics are parameterized by $\mathbf{P}(S^2(\mathcal{S}^*))$, a $\mathbb{P}^5$-bundle on $\mathcal{G}_{2,n}$ of total dimension $3n-1$.

If $C \subset \mathbb{P}^n$ is a smooth conic, it has a quadratic parameterization $g \colon \mathbb{P}^1 \to C$ unique up to automorphism of $\mathbb{P}^1$.

Theorem 1. *Let E be a rank r semistable bundle on $\mathbb{P}^n$ and let $E_C = \bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_j)$ be the generic splitting of E on a smooth conic C in $\mathbb{P}^n$, with $a_1 \geq a_2 \geq \cdots a_r$. Assume that either $n = 2$ and r arbitrary or $r = 2$ and n arbitrary. Then $a_j - a_{j+1} \leq 1$ for all j.*

Proof. The proof of the corresponding (Grauert-Mulich) theorem for restrictions to lines [16, Ch.2, Sec.2.1] works in our case with an additional calculation which follows along with a sketch of the proof. First consider the $n = 2$ case. We use the incidence variety $\mathbf{I} = \{(x, \xi) \in \mathbb{P}^2 \times \mathbb{P}^5 \mid \xi(x) = 0\}$ and the projections

$$\begin{array}{ccc} \mathbf{I} & \xrightarrow{\pi_1} & \mathbb{P}^5 \\ {\scriptstyle \pi_0}\downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

Suppose $a_i - a_{i+1} \geq 2$ for some i. We will obtain a contradiction to E being semistable by constructing a rank i reflexive subsheaf E' of E such that $i^{-1} c_1(E') \cdot \omega_0 > r^{-1} c_1(E) \cdot \omega_0$. This is done by first constructing a rank i reflexive subsheaf $\hat{E}'$ of $\pi_0^* E$ on $\mathbf{I}$ determined as follows. Set $U \equiv \{\xi \in \mathbb{P}^5 \mid C_\xi$ smooth and E_ξ splits generically $\}$. For every $\xi \in U$, $\hat{E}'_{\pi_1^{-1}(\xi)} \equiv \bigoplus_{j=1}^{i} \mathcal{O}(a_j)$. The quotient sheaf $\hat{E}'' \equiv \pi_0^* E / \hat{E}'$ satisfies, for each $\xi \in U$, $\hat{E}''_{\pi_1^{-1}(\xi)} \cong \bigoplus_{j=i+1}^{n} \mathcal{O}(a_j)$. By the descent lemma [16, Chap.2, Lemma 2.1.2], there will be a subsheaf E' of E on $\mathbb{P}^2$ such that $\pi_0^* E' = \hat{E}'$ if $h^0(\mathbf{I}, \mathcal{H}om(T_{\mathbf{I}|\mathbb{P}^2}, \mathcal{H}om(\hat{E}', \hat{E}''))) = 0$ where $T_{\mathbf{I}|\mathbb{P}^2}$ is the bundle of vectors tangent to the fibers of π_0. This E' clearly yields the desired contradiction. So we must calculate $T_{\mathbf{I}|\mathbb{P}^2}$.

The variety $\mathbf{I} \subset \mathbb{P}^2 \times \mathbb{P}^5$ is defined as the zero set of the homogeneous polynomial of bidegree $(2,1)$, $f(x, \xi) = x^t \xi x$ so $(v, \eta) \in T\mathbb{P}^2 \oplus T\mathbb{P}^5$ is tangent to $\mathbf{I}$ at (x, ξ) if and only if $v^t \xi x + x^t \xi v + x^t \eta x = 0$. A tangent to the fiber of π_0 at (x, ξ) has the form $(0, \eta)$ satisfying $x^t \eta x = 0$. The Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5} \xrightarrow[\xi]{} S^2 V^* \otimes \mathcal{O}_{\mathbb{P}^5}(1) \longrightarrow T\mathbb{P}^5 \longrightarrow 0$$

shows that $T\mathbb{P}^5_\xi \cong S^2 V^* / \mathbf{C}\xi$ so $(T_{\mathbf{I}|\mathbb{P}^2})_{x,\xi} \cong \{\eta \in S^2 V^* / \mathbf{C}\xi \mid x^t \eta x = 0\}$.

Set $\tilde{C}_\xi \equiv \pi_1^{-1}(\xi)$ for $\xi \in U$, the copy of the conic C_ξ in the incidence variety. To show $\mathcal{H}om(T_{\mathbf{I}|\mathbb{P}^2}, \mathcal{H}om(\hat{E}', \hat{E}''))$ has no global sections it is enough to show it has no sections along $\tilde{C}_\xi$. Let $g \colon \mathbb{P}^1 \to \tilde{C}_\xi \subset \mathbf{I}$ be a quadratic parameterization. Pulling back $T_{\mathbf{I}|\mathbb{P}^2}$ by g we get over $\mathbb{P}^1$, with homogeneous coordinates $u = (u_0, u_1)$,

$$0 \longrightarrow g^*T_{\mathbf{I}|\mathbb{P}^2} \longrightarrow S^2V^*/\mathbf{C}\xi \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{B} \mathcal{O}_{\mathbb{P}^1}(4) \longrightarrow 0$$

where $B(u)(\eta) \equiv b(u) \cdot \eta$ for $b(u)$ a six-dimensional vector function whose components are homogeneous polynomials of degree 4. Concluding that $g^*T_{\mathbf{I}|\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 4}$ is therefore equivalent to showing that the map induced by B on sections, $S^2V^*/\mathbf{C}\xi \to H^0(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(4))$,is an isomorphism. This must be true or else there is a $\xi' \in S^2V^*$ linearly independent of ξ such that $C_\xi \subset C_{\xi'}$, which is absurd. Now we have

$$\mathcal{H}om(T_{\mathbf{I}|\mathbb{P}^2}, \mathcal{H}om(\hat{E}', \hat{E}''))_{\tilde{C}_\xi} \cong \bigoplus_{j \le i, j' > i} \mathcal{O}_{\mathbb{P}^1}(-a_j + a_{j'} + 1)^{\oplus 4}.$$

Since $a_j \ge a_{j'} + 2$ for all j, j', there are no sections. This proves our result for bundles over $\mathbb{P}^2$.

For E on $\mathbb{P}^n$ of rank 2 and $n \ge 3$, Maruyama proved [14] (see also [8, Sec. 3.3] and [9, Sec.3.2]) that E_P is semistable for the generic plane P in $\mathbb{P}^n$. Now using the incidence variety

$$\begin{array}{ccc} \mathbf{I} & \xrightarrow{\pi_1} & \mathbf{P}(S^2\mathcal{S}^*) \\ \Big\downarrow{\scriptstyle \pi_0} & & \\ \mathbb{P}^n & & \end{array}$$

the same proof works. The same calculation is done on $\tilde{C}_{\xi,P} \equiv \pi_1^{-1}(\xi, P)$ for $P \equiv \mathbb{P}^2 \subset \mathbb{P}^n$ such that E_P is semistable and ξ is a smooth conic in P on which E_P splits generically. A tangent vector to the fiber of π_0 now has the form (η, t) for $\eta \in S^2\mathcal{S}_P^*/\mathbf{C}\xi$ such that $x^t\eta x = 0$ and $t \in (T\mathcal{G}_{2,n})_P$.

$$g^*T_{\mathbf{I}|\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3(n-2)}.$$

$\square$

Corollary 1. *Let E be a rank 2 bundle on $\mathbb{P}^n$ normalized so that $c_1 = 0$ or -1. Then E is semistable if and only if for C a generic smooth conic in $\mathbb{P}^n$ and for $\mathbb{P}^1 \to C$ a quadratic parameterization,*

$$E_C \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} & \text{if } c_1 = 0 \\ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & \text{if } c_1 = -1. \end{cases}$$

Proof. If E is semistable, the splitting follows from Theorem 1. Conversely, if E has the indicated generic splitting on conics and $\mathcal{O}_{\mathbb{P}^n}(k) \to E$ is a non-zero bundle map then restricting to a generic conic implies that $k \le 0$ if $c_1 = 0$ and $k \le -1$ if $c_1 = -1$. $\square$

In connection with Theorem 2, recall that indecomposable rank 2 bundles are plentiful on $\mathbb{P}^2$ and on $\mathbb{P}^3$ [1], [8] but that the only known examples on $\mathbb{P}^4$ are variants of the Horrocks-Mumford bundle [11]. Furthermore, there are no known examples on $\mathbb{P}^n$ for $n \ge 5$ and the Hartshorne conjecture predicts that none exist (at least for $n \ge 7$;see [16]).

For $r \ge 3, n \ge 3$ there are semistable rank r bundles on $\mathbb{P}^n$ whose restrictions to all planes are not semistable (see [3]), e.g. $T\mathbb{P}^3$.

4. Jumping Conics

Let E be a rank 2 semistable bundle on $\mathbb{P}^2$ normalized so that $c_1(E) = 0$ or -1.We want to define a jumping conic C for E. If C is smooth (equivalently irreducible), this is clear in the light of Corollary 1: If $c_1(E) = 0$, $E_C = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$ and, if $c_1(E) = -1$, $E_C = \mathcal{O}_{\mathbb{P}^1}(a-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a-1)$. In both cases, we define C to be a jumping conic for E if $a > 0$. We give a separate definition for singular (equivalently reducible) conics $C = L_1 + L_2$ when $c_1(E) = 0$:C is jumping if either L_1 or L_2 is a jumping line. In the $c_1(E) = -1$ case, for C smooth or singular, define C to be a jumping conic if $h^0(C; E_C) > 0$. Note that this agrees with our previous definition when C is smooth. When C is singular, C is jumping exactly when either: i)L_1 or L_2 is a jumping line, ii)$C = 2L$ and L is a jumping line of the second kind [10], or iii)L_1 and L_2 are generic so that $E_{L_j} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ $j = 1, 2$ and the $\mathcal{O}_{\mathbb{P}^1}$ summands coincide at $p = L_1 \cap L_2$.

Denote the set of jumping conics by J_2. The virtue of these definitions is demonstrated by showing that J_2 can be given a scheme structure in a natural way.

Theorem 2. *The set of jumping conics J_2 of a rank 2 semistable bundle E on $\mathbb{P}^2$ can be given the scheme structure of a hypersurface in $\mathbb{P}^5$ of degree $c_2(E)$ if $c_1(E) = 0$ and of degree $c_2(E) - 1$ if $c_1(E) = -1$. Furthermore the singular jumping conics are in the scheme-theoretic closure of the smooth jumping conics.*

Proof. (The case $c_1(E) = -1$.) For $C \subset \mathbb{P}^2$ any conic, Riemann-Roch and the sequence

$$0 \longrightarrow E(k-2) \longrightarrow E(k) \longrightarrow E_C(k) \longrightarrow 0 \tag{4.1}$$

imply $\chi(C; E_C(k)) = \chi(\mathbb{P}^2; E(k)) - \chi(\mathbb{P}^2; E(k-2)) = 4k$. This shows that $\pi_0^* E$ is flat over $\mathbb{P}^5$ and that $h^0(C; E_C) = h^1(C; E_C)$. Therefore C is a jumping conic if and only if $h^1(C; E_C) > 0$.

The cohomology sequence of (4.1) for $k = 0$ gives

$$0 \to H^0(C; E_C) \to H^1(\mathbb{P}^2; E(-2)) \xrightarrow{f_C} H^1(\mathbb{P}^2; E) \to H^1(C; E_C) \to 0. \tag{4.2}$$

From Riemann-Roch and stability, $h^1(\mathbb{P}^2; E(-2)) = h^1(\mathbb{P}^2; E) = c_2(E) - 1$. Therefore C is a jumping conic if and only if $\det f_C = 0$.

Now consider the incidence diagram

$$\begin{array}{ccc} \mathbf{I} & \xrightarrow{\pi_1} & \mathbb{P}^5 \\ {\scriptstyle \pi_0}\downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

which is the restriction to $\mathbf{I}$ of

$$\begin{array}{ccc} \mathbb{P}^2 \times \mathbb{P}^5 & \xrightarrow{p_1} & \mathbb{P}^5 \\ {\scriptstyle p_0}\downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

Note that $E_0 \equiv \pi_{1*}\pi_0^* E = 0$. Set $E_1 \equiv R^1_{\pi_1 *}\pi_0^* E$. Taking the direct image via p_1 of the exact sequence on $\mathbb{P}^2 \times \mathbb{P}^5$

$$0 \longrightarrow p_0^*E(-2,-1) \xrightarrow{f} p_0^*E \longrightarrow \pi_0^*E \longrightarrow 0$$

gives on $\mathbb{P}^5$

$$0 \longrightarrow H^1(\mathbb{P}^2;E(-2))\otimes\mathcal{O}_{\mathbb{P}^5}(-1) \xrightarrow{f_1} H^1(\mathbb{P}^2;E)\otimes\mathcal{O}_{\mathbb{P}^5} \longrightarrow E_1 \longrightarrow 0.$$

It follows that $J_2 = sptE_1 = V(\det f_1)$ as sets and so $\det f_1 = 0$ defines a scheme structure on J_2, that of a hypersurface in $\mathbb{P}^5$ of degree $c_2 - 1$.

Finally, we verify that the smooth jumping conics are dense in J_2. The only way this could fail is if the cubic hypersurface of reducible conics $S \subset \mathbb{P}^5$, an irreducible variety, is an irreducible component of J_2 (we have seen above that J_2 has pure dimension 4, i.e. has no lower dimension components). But this would contradict Hulek's result [10] that not all lines are jumping lines of the second kind. □

If $c_1(E) = 0$, the structure of J_2 cannot be defined as above because for C smooth and $E_C \cong \mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(-a)$, C is jumping iff $a > 0$ but $h^1(C;E_C) > 0$ iff $a \geq 2$. Furthermore, taking direct images on $\mathbb{P}^5$ as above,

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{P}^2;E)\otimes\mathcal{O}_{\mathbb{P}^5} \longrightarrow E_0 \longrightarrow H^1(\mathbb{P}^2;E(-2))\otimes\mathcal{O}_{\mathbb{P}^5}(-1) \\ \longrightarrow H^1(\mathbb{P}^2;E)\otimes\mathcal{O}_{\mathbb{P}^5} \longrightarrow E_1 \longrightarrow 0. \end{aligned}$$

where E_0 is a rank 2 reflexive sheaf, $h^1(\mathbb{P}^2;E(-2)) = c_2(E)$, and $h^1(\mathbb{P}^2;E) = c_2(E) - 2 + h^0(\mathbb{P}^2;E)$. Therefore $c(E_0) = (1-\omega_1)^{c_2(E)}c(E_1)$ for ω_1 the positive generator of $H^2(\mathbb{P}^5;\mathbf{Z})$. Assume the jump size a is 1 for all smooth $\xi \in J_2$. Further assume that the jump size is also 1 for the reducible jumping conics; we define this to mean that $h^0(L_1+L_2;E_{L_1+L_2}) = 2$. It follows that $E_1 = 0$ [7, Ch.3, Cor.12.9], E_0 is locally free, and $c(E_0) = (1-\omega_1)^{c_2(E)}$. In particular, $c_3(E_0) = -\binom{c_2(E)}{3}$. This must be zero since E_0 is locally free of rank 2 and so $c_2(E) = 0, 1,$ or 2. Therefore

Proposition 1. *Let E be a rank 2 semistable bundle on $\mathbb{P}^2$ with $c_1(E) = 0$ and $c_2(E) \geq 3$. Then E has jumping conics of jump size ≥ 2 and the support of $R^1_{\pi_1 *}\pi_0^*E$ has codimension ≤ 3.*

To study jumping conics in the $c_1(E) = 0$ case we need two tools: 1.Hurtubise's local analysis of families of rank 2 bundles on $\mathbb{P}^1$ [12] and 2.Some results on rank two bundles on ruled surfaces [4, Chap.6].

Let U be an open subset of $\mathbf{C}^n$ containing the origin and let $x = (x_1, x_2, \ldots x_n)$ be the coordinates. Let z be the standard affine coordinate on $\mathbf{C} \subset \mathbb{P}^1$. For E a rank 2 holomorphic bundle on $\mathbb{P}^1 \times U$ and for U small enough, E is trivial on $U_0 \equiv \{(z,x) \mid z \neq \infty\}$ and on $U_1 \equiv \{(z,x) \mid z \neq 0\}$ and therefore determined by a 2 by 2 holomorphic matrix transition function $f(z,x)$ on $U_0 \cap U_1$. Denote by E_x the bundle restricted to $\mathbb{P}^1 \times \{x\}$.

Proposition 2. *([12, Prop. 2.1 and 2.4]) Let $E_0 \cong \mathcal{O}(k_0)\oplus\mathcal{O}(-k_0)$ and let $k \geq k_0 \geq 0$. For U chosen small enough, E has a transition matrix of the form*

$$f(z,x) = \begin{pmatrix} z^k & p(z,x) \\ 0 & z^{-k} \end{pmatrix}$$

for $p(z,x) = \sum_{j=-k+1}^{k-1} a_j(x)z^j$ *[*$k=0$ *implies* $p\equiv 0$*] and* $p(z,0)=0$ *if* $k=k_0$. *Define*

$$\Gamma_p(x) \equiv \begin{pmatrix} a_0(x) & a_1(x) & \cdots & a_{k-1}(x) \\ a_{-1}(x) & a_0(x) & \cdots & a_{k-2}(x) \\ \cdots & \cdots & \cdots & \cdots \\ a_{-k+1}(x) & \cdots & \cdots & a_0(x) \end{pmatrix}$$

Then $E_x \cong \mathcal{O}(a)\oplus\mathcal{O}(-a)$ *iff rank* $\Gamma_p(x) = k-a$.

Think of E as a family of rank 2 bundles on $\mathbb{P}^1$ parameterized by $x\in U$. If E is trivial on the generic $\mathbb{P}^1$, the jumping lines can then be given a scheme structure as a hypersurface in U, $J=\{x\in U \mid det\Gamma_p(x)=0\}$.

Barth [2, Sect.6]gives a different definition of the jumping line scheme. Applying this to the situation above, take a resolution of $E(-1)$ on $\mathbb{P}^1\times U$ of the form

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E(-1) \longrightarrow 0$$

where $F_0 = \bigoplus_{i=1}^{r+2} \pi_1^*\mathcal{O}_{\mathbb{P}^1}(k_i)$ for $k_i<0$ for each i, $\pi_j, j=1,2$ the projections from $\mathbb{P}^1\times U$, and F_1 locally free of rank r. Taking direct images on U,

$$0 \longrightarrow R^1_{\pi_{2*}}F_1 \xrightarrow{\lambda} R^1_{\pi_{2*}}F_0 \longrightarrow R^1_{\pi_{2*}}E(-1) \longrightarrow 0.$$

with the first two terms being bundles of the same rank and the support of $R^1_{\pi_{2*}}E(-1)$ being the jumping locus J. Thus the scheme structure of J can be defined by $det\lambda=0$ and Barth shows that this is independent of the resolution. Furthermore, Hurtubise proves that these two definitions of the scheme structure agree - this also proves that the Hurtubise definition is independent of the trivializations used and gives a second proof that the Barth definition is independent of the resolution.

Proof. (The $c_1(E)=0$ case.) We now define the hypersurface structure of J_2 when $c_1(E)=0$. It is enough to do this locally in $\mathbb{P}^5$. For $\xi\in\mathbb{P}^5\setminus S$, i.e. for C_ξ a smooth conic, use the Hurtubise definition above in a neighborhood of ξ. This defines the jumping locus $J_2'\subset\mathbb{P}^5\setminus S$ as a complex analytic scheme. We will show that J_2' is quasi-projective by showing that, as a set, J_2' is contained in a closed hypersurface in $\mathbb{P}^5$. Therefore the scheme-theoretic closure of J_2' defines a closed sub-scheme of $\mathbb{P}^5$. J_2 and this closure are shown to be equal as sets, thereby defining a scheme structure on J_2.

Consider the second symmetric power of E, S^2E, and note that $c_1(S^2E)=0$ and $c_2(S^2E)=4c_2(E)$. For C any conic,

$$(4.3)\quad 0\longrightarrow H^0(\mathbb{P}^2;S^2E)\longrightarrow H^0(C;S^2E_C)\longrightarrow$$
$$H^1(\mathbb{P}^2;S^2E(-2))\xrightarrow{\alpha_C} H^1(\mathbb{P}^2;S^2E)\longrightarrow H^1(C;S^2E_C)\longrightarrow 0.$$

since $H^0(\mathbb{P}^2;S^2E(-2))=0$ and $H^2(\mathbb{P}^2;S^2E(-2))\cong H^0(\mathbb{P}^2;S^2E(-1))^*=0$ follow from the semistability of E. From Riemann-Roch, $h^1(\mathbb{P}^2;S^2E(-2))=4c_2(E)$, $h^1(\mathbb{P}^2;S^2E)=4c_2(E)-3+h^0(\mathbb{P}^2;S^2E)$, and $h^1(C;S^2E_C)=h^0(C;S^2E_C)-3$.

We show that

$$(4.4)\qquad C \text{ is a jumping conic iff } h^1(C;S^2E_C)>0.$$

If C is smooth, $E_C \equiv \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$ implies that $h^0(C; S^2E_C)$ equals $2a+2$ if $a \geq 1$ and 3 if $a = 0$ and therefore $h^1(C; S^2E_C)$ equals $2a-1$ if $a \geq 1$ and 0 if $a = 0$. For a singular conic $C = L_1 + L_2$, $h^0(C; S^2E_C) \geq 4$ iff L_1 or L_2 is a jumping line is also checked by direct calculation.

Pulling back S^2E to $\mathbb{P}^2 \times \mathbb{P}^5$ and to $\mathbf{I}$ we have

$$0 \longrightarrow p_0^* S^2E(-2,-1) \longrightarrow p_0^* S^2E \longrightarrow \pi_0^* S^2E \longrightarrow 0.$$

Taking direct images by p_1 gives on $\mathbb{P}^5$

$$\begin{aligned}(4.5)\quad & 0 \longrightarrow H^0(\mathbb{P}^2; S^2E) \otimes \mathcal{O}_{\mathbb{P}^5} \longrightarrow \pi_{1*}\pi_0^* S^2E \longrightarrow \\ & H^1(\mathbb{P}^2; S^2E(-2)) \otimes \mathcal{O}_{\mathbb{P}^5}(-1) \xrightarrow{\alpha} H^1(\mathbb{P}^2; S^2E) \otimes \mathcal{O}_{\mathbb{P}^5} \longrightarrow R^1_{\pi_1 *}\pi_0^* S^2E \longrightarrow 0.\end{aligned}$$

Let $\mathcal{J}$ denote the sheaf $R^1_{\pi_1 *}\pi_0^* S^2E$ and set $Y = spt\mathcal{J}$. Because $\mathcal{J}$ is the cokernel of the bundle homomorphism α and because of (4.3) and (4.4), $\xi \in Y$ iff α_ξ is not surjective iff $h^1(C_\xi; S^2E_{C_\xi}) > 0$ iff $\xi \in J_2$. Therefore $J_2 = Y$ as sets and so J_2' is quasi-projective. We define the scheme structure of J_2 to be the closure of J_2' rather than the scheme structure of Y for two reasons: 1.The definition of J_2' via the method of Hurtubise is more geometrically appealing and 2.The Hurtubise method allows us to calculate the degree of J_2 and to show that the singular jumping conics are in the closure of the smooth ones.

We will show that the degree of the hypersurface $J_2 \subset \mathbb{P}^5$ is $c_2(E)$ by proving that, for the generic line Λ, $\Lambda \cdot J_2 = c_2(E)$. Recall that the reducible (non-smooth) conics form a cubic hypersurface $S \subset \mathbb{P}^5$ and that the singular jumping conics form a proper subset of dimension 3. Therefore the generic line Λ misses the singular jumping conics and so E is trivial on the three reducible conics of $\Lambda \cong \mathbb{P}^1$. Let t be an affine parameter on Λ and set $X \equiv \pi_1^{-1}\Lambda$. Then $X \to \Lambda$ is a ruled surface with three non-smooth fibers C_{t_i} $i = 1,2,3$. Another way to view X is as follows. Λ is a pencil of conics in $\mathbb{P}^2$. Let C_0 and C_∞ be two smooth conics with $C_0 \cap C_\infty = \{p_0, p_1, p_2, p_3\}$ the base locus of Λ. X is $\mathbb{P}^2$ blown up at these four points and C_{t_i} is the proper transform of $\overline{p_0p_i} + \overline{p_jp_k}$ for (i,j,k) a cyclic permutation of $(1,2,3)$. This description implies that $\pi_0^*E|_X$ has $c_1 = 0$ and $c_2 = c_2(E)$. For simplicity, denote $\pi_0^*E|_X$ by E. Note that the ruled surface X can be taken to be smooth because a local calculation shows that smoothness is equivalent to the pencil Λ having four distinct base points.

For all but a finite number of $t \in \Lambda$, $E_{C_t} = \overset{2}{\oplus}\mathcal{O}_{C_t}$. For $t \in \Lambda \cap J_2$, $E_{C_t} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$ for $a \in \mathbf{Z}^+$. Define the rank 2 bundle E' on X as the elementary modification

$$0 \longrightarrow E' \longrightarrow E \xrightarrow{\psi} j_{t*}\mathcal{O}_{\mathbb{P}^1}(-a) \longrightarrow 0$$

for $j_t\colon \mathbb{P}^1 \cong C_t \hookrightarrow X$ and $\psi(\sigma) \equiv \mathcal{O}_{\mathbb{P}^1}(-a)$-component of $\sigma|_{C_t}$. By (2.4), $c_1(E') = -[F]^*$ and $c_2(E') = c_2(E) - a$ (where F is the fiber of $X \to \Lambda$).

We use some of the basic theory of rank 2 bundles over ruled surfaces as developed in [4, Ch.6]. Some of the results we quote are proven for geometrically ruled surfaces (all fibers smooth) but remain true in our case. For E' as above, $E'_{C_t} = \mathcal{O}_{\mathbb{P}^1}(a') \oplus \mathcal{O}_{\mathbb{P}^1}(-a')$ for $0 \leq a' \leq a$ [4, page 151]. For any E trivial on the generic fiber of $\pi_1\colon X \to \Lambda$, $c_2(E) \geq 0$ with equality iff E is trivial on every fiber, i.e. $E = \pi_1^*W$ for W a rank 2 bundle on Λ [4, Ch.6,Thm.10]. Therefore after finitely many elementary

modifications of the type described above (more than one modification may have to be done on the same fiber), we get

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \mathcal{T} \longrightarrow 0$$

for $E' = \pi_1^* W$, W a rank 2 bundle on Λ, and $\mathcal{T}$ a torsion sheaf with support on the jumping fibers. In addition, $c_2(E) = \sum_j a_j$ where the $a_j \in \mathbf{Z}^+$ come from the modifications.

Denote the intersection of Λ and J_2 at t by $(\Lambda \cdot J_2)_t$. We show that $(\Lambda \cdot J_2)_t =$ the sum of the a_j's coming from the elementary modifications on the fiber C_t. This will prove $\Lambda \cdot J_2 = c_2(E)$. For simplicity set $t = 0$ and set U be a small disc about 0 in $\mathbb{P}^1$ with coordinate x. If $E_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$, apply Proposition 2 with $k = k_0 = a$ so that E on $\pi_1^{-1}(U)$ is described by the transition function

$$f(z, x) = \begin{pmatrix} z^a & p(z, x) \\ 0 & z^{-a} \end{pmatrix}$$

for $p(z, x) = \sum_{j=-a+1}^{a-1} a_j(x) z^j$ and $a_j(0) = 0$ for each j. Write $a_j(x) = x b_j(x)$ and $p(z, x) = x q(z, x)$. If $\mathcal{O} \oplus \mathcal{O}$ is the trivialization of E on $U_0 = \{(z, x) \mid z \neq \infty\}$, the definition of the elementary modification E' implies that the corresponding trivialization of E' is $x\mathcal{O} \oplus \mathcal{O} \subset \mathcal{O} \oplus \mathcal{O}$. The transition function for E' is therefore

$$\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z^a & p(z, x) \\ 0 & z^{-a} \end{pmatrix} \cdot \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z^a & q(z, x) \\ 0 & z^{-a} \end{pmatrix}.$$

Note that $\det \Gamma_p(x) = x^a \det \Gamma_q(x)$. If E'_{C_0} is trivial, $\det \Gamma_q(0) \neq 0$ and $(\Lambda \cdot J_2)_t = a$. If $E'_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_1)$ for $a_1 \in \mathbf{Z}^+$, then $a_1 \leq a$ and we apply Proposition 2 again with $k = k_0 = a_1$. We get another transition function for E' of the form

$$\begin{pmatrix} z^{a_1} & r(z, x) \\ 0 & z^{-a_1} \end{pmatrix}$$

for $r(z, x) = \sum_{j=-a_1+1}^{a_1-1} \alpha_j(x) z^j$ and $\alpha_j(0) = 0$ for each j. Because Hurtubise's definition is independent of the trivialization used, $det\Gamma_q(x)$ and $det\Gamma_r(x)$ have the same order zero at $x = 0$. $det\Gamma_r(x) = x^{a_1} det\Gamma_s(x)$ for $r(z, x) = xs(z, x)$. Therefore the order of the zero of $det\Gamma_p(x)$ at $x = 0$ is $a + a_1 +$ order of $det\Gamma_s(x)$ at $x = 0$. Our result follows by induction.

Finally, we prove that $\overline{J_2'} = J_2$. For $\psi \colon J_1 \times \mathbb{P}^{2*} \to S \subset \mathbb{P}^5$ defined by $\psi(l, l') = l \cdot l'$, the image of ψ is $J_2 \cap S$. Let $J_1 = \sum_\mu m_\mu J_{1\mu}$ as a divisor; $J_{1\mu}$ is an irreducible component of J_1 for each μ. Then the irreducible components of $S \cap J_2$ are the 3-dimensional varieties $Y_\mu \equiv \psi(J_{1\mu} \times \mathbb{P}^{2*})$. Suppose $\eta \in Y_\mu$ is not in $\overline{J_2'}$. Since $\psi(J_1 \times J_1)$ is 2-dimensional, we can assume $\eta \notin \psi(J_1 \times J_1)$, that is, $C_\eta = L_1 + L_2$ for L_1 a generic line and L_2 a jumping line. Since the set of conics tangent to C_η at some point is 4-dimensional, the generic choice of $\xi \in \mathbb{P}^5$ corresponds to a smooth E-generic conic not tangent to C_η. Letting Λ be the line in $\mathbb{P}^5$ through η and ξ, we can also assume $\Lambda \cap (\overline{J_2'} \cap S) = \emptyset$. Let $X = \pi_1^{-1}\Lambda$ be the corresponding ruled surface and denote $\pi_0^* E|_X$ by E as above. The jumping fibers of $X \to \Lambda$ include C_η and the fibers above the points of $\overline{J_2'} \cap \Lambda$, the latter being smooth. From $C_\eta^2 = 0$ and $C_\eta \cdot K_X = -2$ it follows that $L_i^2 = -1$ and $L_i \cdot K_X = -1$ for $i = 1, 2$. Blow down the generic line L_1 to produce a ruled surface $\check{X}$. L_2 blows down to a smooth fiber $\check{L}_2$ of $\check{X}$. By a theorem of Schwarzenberger [18, Thm.5], the bundle E descends to a

bundle $\check{E}$ on $\check{X}$ ($E = \pi^*\check{E}$ for $\pi\colon X \to \check{X}$ the blow-down map) because E is trivial L_1. Note that $c_2(\check{E}) = c_2(E)$, $c_1(\check{E}) = c_1(E)$, and $\check{E}_{\check{L}_2} \cong \mathcal{O}_{\check{L}_2}(b) \oplus \mathcal{O}_{\check{L}_2}(-b)$ for $b \in \mathbf{Z}^+$ (if $b = 0, E_{L_2} = \pi^*\check{E}_{\check{L}_2}$ would be trivial, a contradiction). Also note that the splitting of E on C_t is the same as that of $\check{E}$ on $\check{C}_t = \pi(C_t)$ for all $t \neq \eta$.

Our previous calculation shows that, for each $t \in \Lambda$ such that C_t is smooth, $(\Lambda \cdot \overline{J_2'})_t = \sum a_j$ where the sum is over all $a_j \in \mathbf{Z}^+$ coming from the elementary modifications involving the fiber C_t needed to change E so that it becomes trivial on C_t. We used this to prove that $\Lambda \cdot \overline{J_2'} = c_2(E)$ for the generic Λ. Since the Λ chosen above is not contained in $\overline{J_2'}$, $\Lambda \cdot \overline{J_2'} = c_2(E)$. Doing the same calculation for $\check{E}$ gives

$$\begin{aligned} c_2(E) &= c_2(\check{E}) \\ &\geq b + \sum_{t \in \Lambda \cap \overline{J_2'}} (\Lambda \cdot \overline{J_2'})_t \\ &= b + \Lambda \cdot \overline{J_2'} \\ &= b + c_2(E) \end{aligned}$$

which is a contradiction. Therefore $\overline{J_2'} = J_2$. $\square$

Corollary 2. *Let E be a stable rank 2 bundle on $\mathbb{P}^2$ with $c_1(E) = 0$. Let C be a smooth jumping conic of E of jump size a corresponding to $\xi \in J_2 \subset \mathbb{P}^5$. Let Λ be a line in $\mathbb{P}^5$ through ξ not contained in J_2. Then the intersection multiplicity of Λ and J_2 at ξ satisfies $(\Lambda \cdot J_2)_\xi \geq a$.*

5. Rank 2 Stable Bundles on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 2$

Let E be a rank 2 stable bundle on $\mathbb{P}^2$ with $c_1 = 0$ and $c_2 = 2$. Set $W \equiv H^0(\mathbb{P}^2; E(1))$ and note that $\sigma \in W$ has 3 zeros. The sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\ \sigma\ } E \xrightarrow{\ \sigma\wedge\ } \mathcal{I}_{Z_\sigma}(1) \longrightarrow 0$$

and the stability of E imply that the zeros of σ are not colinear. Friedman [4] shows that dim $W = 4$ and that E can be described by a sequence

$$(5.1) \qquad 0 \longrightarrow W' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\ f\ } W \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\ e\ } E(1) \longrightarrow 0$$

where e is evaluation of sections and f is a 4-by-2 matrix of homogeneous degree 1 polynomials. Denote the two columns of $f(x)$ by $f_j(x)$, $j = 1, 2$. We will also use f to denote the induced map $f\colon \mathbb{P}^2 \to \mathbf{G}$ ($\mathbf{G}$ is the grassmannian of 2-planes through $0 \in W$ or the lines in $\mathbf{P}W \cong \mathbb{P}^3$) and write $f(x) = f_1(x) \wedge f_2(x)$. $f(x) = \{\sigma \in W \mid \sigma(x) = 0\}$. Note that $E(1) \cong f^*\mathcal{Q}$ for $\mathcal{Q}$ the standard rank-2 quotient bundle on $\mathbf{G}$ so that f determines E. Also notice that $f^*\mathcal{Q} \cong g^*\mathcal{Q}$ iff there is an $A \in GL_4$ and a $B \in SL_2$ such that $g = AfB$. The parameter count $dim\{f\} - dim\{A\} - dim\{B\} = 8 \cdot 3 - 16 - 3 = 5$ agrees with dim $\mathcal{M}(0, 2)$. We will describe the jumping lines and conics of E in terms of the map f.

First recall the Schubert cycles whose classes generate the cohomology of $\mathbf{G}$ [5, Ch.1,Sec.5]. Let $\Lambda \subset \mathbf{P}W$ be the line corresponding to $l \in \mathbf{G}$. A line $\Lambda_0 \subset \mathbf{P}W$ defines the codimension one cycle $Z_1(\Lambda_0) \equiv \{l \in \mathbf{G} \mid \Lambda_0 \cap \Lambda \neq \emptyset\}$; its dual cohomology class is given by the Fubini-Study form ω_1. A point $w \in \mathbf{P}W$ defines

the codimension two cycle $Z_2(w) \equiv \{l \in \mathbf{G} \mid w \in \Lambda\}$; its dual class is denoted ω_2. A 2-plane $P \subset \mathbf{P}W$ defines the codimension two cycle $Z_{1,1}(P) \equiv \{l \in \mathbf{G} \mid \Lambda \subset P\}$; its dual class is denoted $\omega_{1,1}$. We have

$$\omega_1^2 = \omega_2 + \omega_{1,1} \tag{5.2}$$

and

$$c(\mathcal{Q}) = 1 + \omega_1 + \omega_2. \tag{5.3}$$

It is sometimes useful to regard W as an abstract 4-dimensional vector space and E as being the quotient bundle defined by (5.1). The cohomology sequence of (5.1) then implies that global sections σ have the form $\sigma = \sigma_w$ for $w \in W$ and $\sigma_w(x) = w \bmod f(x)$. Therefore $Z_{\sigma_w} = f^{-1}(Z_2(w))$ and so

$$\int_{\mathbb{P}^2} f^*\omega_2 = 3. \tag{5.4}$$

Since f is quadratic,

$$\int_{\mathbb{P}^2} f^*\omega_1^2 = \int_{\mathbb{P}^2} (2\omega_0)^2 = 4 \tag{5.5}$$

and therefore (5.2) and (5.4)imply

$$\int_{\mathbb{P}^2} f^*\omega_{1,1} = 1 \tag{5.6}$$

and $f(\mathbb{P}^2) \cdot Z_{1,1}(P) = 1$ for the generic 2-plane P.

Let L be a line in $\mathbb{P}^2$ and let p and q be distinct points on L. $E(1)_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(1)$ for L generic and $E(1)_L \cong \mathcal{O}_L(2) \oplus \mathcal{O}_L$ for L a jumping line (any other splitting contradicts $E(1)$ being globally generated).

Proposition 3. *Let E be a rank 2 stable bundle on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 2$ and let $f\colon \mathbb{P}^2 \to \mathbf{G}$ be the corresponding map. The line $L = \overline{pq}$ is a jumping line for E iff $f(p) \wedge f(q) = 0$ iff there is a Schubert cycle $Z_{1,1}(P)$ in $\mathbf{G}$ such that $L = f^{-1}(Z_{1,1}(P))$.*

Proof. Parameterize L by $t_0 p + t_1 q$; then f restricted to L is

$$f_L(t) = (t_0 f_1(p) + t_1 f_1(q)) \wedge (t_0 f_2(p) + t_1 f_2(q)).$$

We collect some facts about f. f is 1-to-1 because if $f(p) = f(q) = u \wedge v \in \mathbf{G}$, f is constant on $\overline{pq}$ and so σ_u vanishes on $\overline{pq}$ contradicting the stability of E. The image of f does not lie in any cycle $Z_1(\Lambda_0)$ nor therefore in any cycle $Z_{1,1}(P)$. This holds because if $f(\mathbb{P}^2) \subset Z_1(\Lambda_0)$, $\mathbb{P}^2 \subset \bigcup_{w \in \Lambda_0} Z_{\sigma_w}$ which is impossible by a dimension count.

If $f(p) \wedge f(q) \neq 0$, $f_1(p)$, $f_1(q)$, $f_2(p)$, $f_2(q)$ form a basis for W. It follows that $\sigma \in W$ can have at most one zero on L which implies that L is generic.

If $f(p) \wedge f(q) = 0$, the span of $f_1(p)$, $f_1(q)$, $f_2(p)$, $f_2(q)$ has dimension 3 (because f is 1-to-1). Choose a basis w_i, $i = 0$ to 3 for W so that

$$f_L(t) = (t_0 w_0 + t_1 w_1) \wedge (t_0 w_2 + t_1 [a w_0 + b w_1 + c w_2])$$

It follows that σ_{w_2} has two zeros on L if $b \neq 0$ and σ_{w_0} has two zeros on L if $b = 0$ and so L is a jumping line. Furthermore $L = f^{-1}(Z_{1,1}(P))$ where P is the $\mathbb{P}^2$ spanned by w_i, $i = 0, 1, 2$. Containment is clear. If $r \notin L$ satisfies $f(r) \in Z_{1,1}(P)$, then $f(\mathbb{P}^2) \subset Z_{1,1}(P)$ which we know is impossible. $\square$

Proposition 4. *Let E be a rank 2 stable bundle on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 2$ and let $f\colon \mathbb{P}^2 \to \mathbf{G}$ be the corresponding map. For a conic $C \subset \mathbb{P}^2$, the following are equivalent:*
i) C is a jumping conic for E.
ii) There are global sections σ and s of $E(1)$ such that $C = Z_{\sigma\wedge s}$.
iii) There is a Schubert cycle $Z_1(\Lambda_0) \subset \mathbf{G}$ such that $C = f^{-1}(Z_1(\Lambda_0))$.

Proof. Let C be a smooth conic and give it a quadratic parameterization $\mathbb{P}^1 \to C$. $E(1)_C \cong \mathcal{O}_{\mathbb{P}^1}(2+a) \oplus \mathcal{O}_{\mathbb{P}^1}(2-a)$ and C is jumping iff $a \geq 1$. Set $W_1 \equiv \{\sigma \in W \mid \mathcal{O}_{\mathbb{P}^1}(2-a) - \text{component of } \sigma = 0\}$. If C is a jumping conic, dim $W_1 \geq 2$. For σ, $s \in W_1$ independent, $C = Z_{\sigma\wedge s}$. Conversely, if $C = Z_{\sigma\wedge s}$, σ and s each have all three of their zeros on C so $a = 1$ and C is a jumping conic. Recall that σ has the form σ_u for $u \in W$ and $\sigma_u(x) = u \bmod f(x)$ and, similarly, $s = s_v$. Thus the homogeneous equation of degree 2 for a jumping conic C can be written $u \wedge v \wedge f(x) = 0$. It follows that $C = f^{-1}(Z_1(u \wedge v))$.

We have shown that the image of the map from $\mathbf{G}$ to J_2 given by $u\wedge v \to Z_{\sigma_u\wedge s_v}$ contains all the smooth jumping conics and therefore the map is surjective. $\square$

6. Rank 2 Stable Bundles on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 3$

Let E be a rank 2 stable bundle on $\mathbb{P}^2$ with $c_1 = 0$ and $c_2 = 3$. From Riemann-Roch, $h^0(\mathbb{P}^2; E(1)) \geq 3$ and, for a non-zero section $\sigma \in H^0(\mathbb{P}^2; E(1))$, the zero set Z_σ is a 0-dimensional subscheme of length 4 and

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\sigma} E(1) \xrightarrow{\sigma\wedge} \mathcal{I}_{Z_\sigma}(2) \longrightarrow 0.$$

Now assume that no three points of Z_σ are collinear. In this case, we say that E is a bundle of general type and prove later that all sections of E have zero sets with no three points collinear. Friedman [4, page 94] points out that $h^0(\mathbb{P}^2; E(1)) = 3$ and that $E(1)$ is globally generated, that is, for $W \equiv H^0(\mathbb{P}^2; E(1))$,

$$(6.1) \qquad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{f} W \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{e} E(1) \longrightarrow 0.$$

where e is evaluation of sections and f is defined by a 3-vector of degree 2 homogeneous polynomials. We also use f to denote the induced regular map

$$(6.2) \qquad f\colon \mathbb{P}^2 \to \mathbf{P}W \cong \mathbb{P}^2,$$

a 4-to-1 branched cover. Note that $f(x)$ equals the point in $\mathbf{P}W$ corresponding to the 1-dimensional space of sections that vanish at x. [If $\sigma \in W \setminus 0$ satisfies $\sigma(x) = 0$ we will also write $f(x) = \sigma$.] For $\mathcal{Q}$ the canonical quotient bundle on $\mathbb{P}^2$, $f^*\mathcal{Q} = E(1)$ so that f determines E and $f^*\mathcal{Q} = g^*\mathcal{Q}$ iff there is an $A \in GL_3$ such that $g = Af$. Counting parameters, $dim\{E\} = dim\{f\} - dim\{A\} = 3\binom{4}{2} - 9 = 9$ which matches dim $\mathcal{M}(0,3)$ [1]. By the Hurwitz formula, the ramification divisor R of f is a cubic curve.

We will describe the jumping lines and conics of E in terms of the map f and its ramification divisor R. Define the 3 by 3 matrix of degree 1 homogeneous polynomials in x_0, x_1, x_2, $F \equiv (f_{i_{x_j}})$ and set $D \equiv detF = f_{x_0} \wedge f_{x_1} \wedge f_{x_2}$. R is defined by $D(x) = 0$. On an affine piece of $\mathbb{P}^2$, for example on $U_0 \equiv \{x \in \mathbb{P}^2 \mid x_0 \neq 0\}$, we use Euler's identity to write D in inhomogeneous form $D = 3f \wedge f_{x_1} \wedge f_{x_2}$ where x_1, x_2 are the affine coordinates obtained by setting $x_0 = 1$.

For $p \in \mathbb{P}^2$ and $f(p) = \sigma \in \mathbf{P}W$, make a change in the homogeneous coordinates in both the domain and range to obtain $p = e_0$ and e_0, e_1, e_2 a basis for W with $e_0 = \sigma$. The global sections e_1 and e_2 form a local frame for $E(1)$ and so $\sigma = \sigma_1 e_1 + \sigma_2 e_2$ near $p = e_0$. From

$$f_0 e_0 + f_1 e_1 + f_2 e_2 = 0$$

it follows that

$$\sigma_i(x) = -f_i(x)/f_0(x)$$

for $i = 1, 2$ near $p = e_0$ and therefore $df_p = -d\sigma_p$. Defining the index of the zero of σ at p to be $i_p(\sigma) \equiv dim\mathcal{O}_{\mathbb{P}^2,p}/(\sigma_1, \sigma_2)_p$, we see that $p \in R$ iff p is a zero of $\sigma = f(p)$ and $i_p(\sigma) \geq 2$.

Now suppose $p \in R$ so that rank $F_p \leq 2$. If rank $F_p = 1$, R is clearly singular at p and $dD_p = 0$, $df_p = 0$, and $d\sigma_p = 0$. If rank $F_p = 2$, further changes in the homogeneous coordinates allows us to assume $f_{x_1}(e_0) = e_1$ and $f_{x_2}(e_0) = 0$. Therefore

$$\begin{aligned} df_{e_0} &= \begin{pmatrix} 10 \\ 00 \end{pmatrix} \\ d\sigma_{e_0} &= \begin{pmatrix} 10 \\ 00 \end{pmatrix}. \end{aligned} \tag{6.3}$$

with respect to the obvious bases and

$$dD_{e_0} = 3f_{2_{x_2x_1}}(e_0)dx_1 + 3f_{2_{x_2x_2}}(e_0)dx_2. \tag{6.4}$$

Thus R is singular at e_0 iff $f_{2_{x_2x_1}}(e_0) = 0$ and $f_{2_{x_2x_2}}(e_0) = 0$. If R is smooth at e_0, $TR_{e_0} = \operatorname{Ker} df_{e_0} = \mathbf{C}e_2$ iff $f_{2_{x_2x_2}}(e_0) = 0$.

Proposition 5. *The ramification curve R is either a smooth cubic, the union of a line and a smooth conic, or the union of three distinct lines. The last case occurs exactly when there is a point $p \in R$ satisfying rank $F_p = 1$.*

Proof. Let p be a singular point of R. If rank $F_p = 1$, then choosing coordinates correctly, we get $p = e_0$, $f(e_0) = e_0$, and $f_{x_1}(e_0) = 0 = f_{x_2}(e_0)$. Write $f_l = \sum_{0 \leq i \leq j \leq 2} a_{lij} x_i x_j$. Imposing the above conditions we obtain

$$f(x) = \begin{pmatrix} x_0^2 + a_{001}x_0x_1 + a_{002}x_0x_2 + \sum_{1 \leq i \leq j \leq 2} a_{0ij} x_i x_j \\ \sum_{1 \leq i \leq j \leq 2} a_{1ij} x_i x_j \\ \sum_{1 \leq i \leq j \leq 2} a_{2ij} x_i x_j \end{pmatrix}$$

My making an additional linear change in x_0, x_1, x_2 we can put f in one of the following two forms:

$$f(x) = \begin{pmatrix} x_0^2 + \sum_{1\le i\le j\le 2} a_{0ij}x_ix_j \\ x_1x_2 \\ (x_2-\alpha x_1)(x_2-\beta x_1) \end{pmatrix}$$

for $\alpha \neq 0$ and $\beta \neq 0$ (this follows from the fact that f has finite fibers) or

$$f(x) = \begin{pmatrix} x_0^2 + \sum_{1\le i\le j\le 2} a_{0ij}x_ix_j \\ x_1^2 \\ x_2^2 \end{pmatrix}.$$

In the first case,

$$D = 4x_0(x_2^2 - \alpha\beta x_1^2)$$

and we see that R is the union of three distinct lines. The second case gives the same conclusion.

If p is a singular point of R and rank $F_p = 2$, choose coordinates as above to obtain

$$f(x) = \begin{pmatrix} x_0^2 + \sum_{1\le i\le j\le 2} a_{0ij}x_ix_j \\ x_0x_1 + \sum_{1\le i\le j\le 2} a_{1ij}x_ix_j \\ a_{211}x_1^2 \end{pmatrix}$$

The fact that $a_{212} = 0$ and $a_{222} = 0$ was shown above. Note that $a_{211} \neq 0$ and $a_{122} \neq 0$ follow from the fact that f has finite fibers. Direct calculation gives

$$D = 2a_{211}x_1[2x_0(2a_{122}x_2 + a_{112}x_1) - x_1(2a_{022}x_2 + a_{012}x_1)]$$

and therefore $R = L + C$ for L the line defined by $x_1 = 0$ and C a conic. The conic is degenerate iff $a_{122}(a_{122}a_{012} - a_{112}a_{022}) = 0$, i.e. $a_{122}a_{012} - a_{112}a_{022} = 0$. In this case, make the linear change of homogeneous coordinates in the range given by the matrix

$$\begin{pmatrix} 1 & 0 & -a_{011}/a_{211} \\ 0 & 1 & -a_{111}/a_{211} \\ 0 & 0 & 1 \end{pmatrix}$$

and then another given by

$$\begin{pmatrix} 1 & -\lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for λ defined by $\lambda(a_{122}, a_{112}) = (a_{022}, a_{012})$. This yields

$$f = \begin{pmatrix} x_0^2 - \lambda x_0x_1 \\ x_0x_1 + a_{112}x_1x_2 + a_{122}x_2^2 \\ a_{211}x_1^2 \end{pmatrix}$$

Calculating F and D shows that R is the union of three distinct lines and that rank $F_{e_2} = 1$. $\square$

For any line $L \subset \mathbb{P}^2$, $E(1)_L \cong \mathcal{O}_L(1+a) \oplus \mathcal{O}_L(1-a)$ with $0 \le a$. For L generic, $a = 0$. Since $E(1)$ is globally generated, $a \le 1$ so the jumping lines correspond to $a = 1$. Note that this means that for every global section σ of $E(1)$, no 3 of the 4

zeroes of σ are collinear. Let f_L be the restriction f to L and denote the image by $\hat{L}$.

If L is generic, $E(1)_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(1)$ implies that f_L is 1-to-1. Since f is quadratic and L is irreducible, $\hat{L}$ is a smooth conic.

For L jumping, $E(1)_L \cong \mathcal{O}_L(2) \oplus \mathcal{O}_L$ implies that f_L is 2-to-1 and that $\hat{L}$ is a line. By the Hurwitz theorem, f_L has 2 ramification points, 2 of the 3 points of $L \cdot R$. We now work in the opposite direction by starting with a point of R and producing a jumping line. For $p \in R$ with rank $d\sigma_p = 1$, let $v \in T\mathbb{P}^2_p$ span $\operatorname{Ker} df_p = \operatorname{Ker} d\sigma_p$. For L the line through p with direction v and $\sigma = f(p)$, σ_L clearly has a double zero at p and so $E(1)_L \cong \mathcal{O}_L(2) \oplus \mathcal{O}_L$. L meets R at 2 more points. One of them, denoted by q, must be distinct from p and satisfy $TL_q = \operatorname{Ker} d\tau_q$ for $\tau = f(q)$. Thus p and q are the ramification points of the 2-to-1 branched cover $f_L \colon L \to \hat{L}$.

Proposition 6. *Let E be a rank 2 stable bundle on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 3$ of general type. Let $f \colon \mathbb{P}^2 \to \mathbb{P}^2$ be the corresponding map with ramification curve R. If R is smooth, there is a regular 2-to-1 map $g \colon R \to J_1$ onto the cubic curve of jumping lines of E.*

Proof. The definition of the jumping line $g(p)$ corresponding to $p \in R$ was given above. Because R is smooth, rank $df_p = 1$. As in the previous proof, we can change homogeneous coordinates in the domain and range so that $p = e_0$, $f(e_0) = e_0$, $f_{x_1}(e_0) = e_1$, and $f_{x_2}(e_0) = 0$. In local coordinates x_1, x_2 near p,

$$f(x) = \begin{pmatrix} 1 \\ f_1(1, x_1, x_2)/f_0(1, x_1, x_2) \\ f_2(1, x_1, x_2)/f_0(1, x_1, x_2) \end{pmatrix}$$

For $x \in R$ near p, the kernel of df_x is spanned by

$$v(x) = \begin{pmatrix} 0 \\ -(f_1/f_0)_{x_2} \\ (f_1/f_0)_{x_1} \end{pmatrix}$$

For $x \in R$ near p, $g(x) = x \wedge v(x)$ and so g is a regular mapping. □

Let $C \subset \mathbb{P}^2$ be a smooth conic. $E(1)_C \cong \mathcal{O}_{\mathbb{P}^1}(2+a) \oplus \mathcal{O}_{\mathbb{P}^1}(2-a)$ for $a \geq 0$. Because E is globally generated, $a \leq 2$. The smooth jumping conics have $a = 1$ or 2. If $a = 2$ and $\xi \in J_2$ is the point corresponding to C, J_2 is singular at ξ by Corollary 2.

Denote the restriction of f to C by f_C and its image by $\hat{C}$.

First consider a smooth generic conic: $E(1)_C \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. For ω_0 the Fubini-Study form on $\mathbb{P}^2$,

$$\int_C f_C^* \omega_0 = \int_C 2\omega_0 = 4$$

so either i) $deg \hat{C} = 1$ and f_C is generically 4-to-1, ii) $deg \hat{C} = 2$ and f_C is generically 2-to-1, or iii) $deg \hat{C} = 4$ and f_C is generically 1-to-1. Case i) can not occur because a section over C can have at most 2 zeroes. Case ii) can not occur or else for (σ_1, σ_2) defined as the image of σ by

$$0 \to H^0(\mathbb{P}^2; E(1)) \xrightarrow{rest.} H^0(C; E(1)_C) \cong \overset{2}{\oplus} H^0(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(2))$$

$\sigma_1 = c\sigma_2$ for $c \in \mathbf{C}^*$ (c could depend on σ). But then, for any $x \in C$, $dim\{\sigma \mid \sigma(x) = 0\} = 2$, which contradicts $E(1)$ being globally generated. Thus $f_C\colon C \to \hat{C}$ is generically 1-to-1 and $deg\hat{C} = 4$. It follows that the arithmetic genus of $\hat{C}$ is 3, that $\hat{C}$ is singular, that f_C is a normalization of $\hat{C}$, and that, for

$$0 \longrightarrow \mathcal{O}_{\hat{C}} \xrightarrow{f_C^*} f_{C*}\mathcal{O}_C \longrightarrow f_{C*}\mathcal{O}_C/\mathcal{O}_{\hat{C}} \longrightarrow 0,$$

the length of the skyscraper sheaf $f_{C*}\mathcal{O}_C/\mathcal{O}_{\hat{C}}$ is 3. Since a section over C can have at most two zeros, there are three singular points in $\hat{C}$, each of whose inverse images consists of exactly two points, counting multiplicity.

For C a smooth jumping conic with $a = 1$, $E(1)_C \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Consider the injective homomorphism r

$$W \equiv H^0(\mathbb{P}^2; E(1)) \overset{rest.}{\to} H^0(C; E(1)_C) \to H^0(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(3)) \oplus H^0(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(1))$$

and write $r(\sigma) = (r_1(\sigma), r_2(\sigma)) = (\sigma_1, \sigma_2)$. $E(1)$ being globally generated implies that r_2 is surjective. Let $\tau \in H^0(\mathbb{P}^2; E(1))$ span $\mathrm{Ker}\, r_2$. If $\sigma \in W$ is a multiple of τ (i.e. $\sigma_2 = 0$), it has three zeros on C. Otherwise the only zero of σ on C is the unique zero of σ_2. Therefore the triple point τ is the lone singular point of $\hat{C}$ and, as before, $\hat{C}$ has degree 4 and $l(f_{C*}\mathcal{O}_C/\mathcal{O}_{\hat{C}}) = 3$.

For a smooth jumping conic with $a = 2$, $E(1)_C \cong \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}$. Clearly $dim\{\sigma \in W \mid \mathcal{O}_{\mathbb{P}^1} - \text{component of } \sigma|_C = 0\} = 2$; let σ, s be a basis. It follows that $C = Z_{\sigma\wedge s}$. Conversely, if σ, s are independent in W, $Z_{\sigma\wedge s}$ is a conic C. It is clear that the zero set of σ, as a point set, lies in C. We want to verify that $l(Z_\sigma) = 4$ implies that $l(Z_{\sigma|_C}) = 4$ and therefore $E(1)_C \cong \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}$. For $p \in Z_\sigma$, choose an affine part $\mathbf{C}^2$ of $\mathbb{P}^2$ containing p and a trivialization of $E(1)$ on $\mathbf{C}^2$ so that $\sigma = (\sigma_1, \sigma_2)$ and $s = (s_1, s_2)$. The quadratic equation for C on $\mathbf{C}^2$ is $g = \sigma_1 s_2 - \sigma_2 s_1 = 0$. The multiplicity of the zero p of σ is defined as $i_p(\sigma) \equiv \dim\, \mathcal{O}_{\mathbb{P}^2,p}/(\sigma_1, \sigma_2)_p$. The multiplicity of $\sigma|_C$ at p is

$$\begin{aligned} i_p &\equiv \dim\, \mathcal{O}_{C,p}/(\sigma_1|_C, \sigma_2|_C)_p \\ &= \dim\, \mathcal{O}_{\mathbb{P}^2,p}/(g)_p \Big/ (\sigma_1, \sigma_2)_p/(g)_p \\ &= \dim\, \mathcal{O}_{\mathbb{P}^2,p}/(\sigma_1, \sigma_2)_p \\ &\equiv i_p(\sigma). \end{aligned}$$

Remark: We have shown that a smooth jumping conic C has jump size $a = 2$ iff $C = Z_{\sigma\wedge s}$ and thus $\hat{C}$ is the line spanned by σ and s in $\mathbf{P}W$ and $f_C\colon C \to \mathbb{P}^1$ is a 4-to-1 branched cover. There are 6 ramification points, counting multiplicity and, as a point set, $R_{f_C} \subset R$. Since $R \cdot C = 6$, this suggests that $R_{f_C} = R \cdot C$ as divisors on C but this has not been proved.

We summarize in

Proposition 7. *Let E be a rank 2 stable bundle on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 3$ of general type and let $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ be the corresponding regular map (see (6.2)). Let f_L and f_C be the restrictions of f to a line L and a smooth conic C respectively. Then*

i) L is generic iff f_L is 1-to-1 onto a smooth conic.

ii) L is a jumping line iff the jump size is 1 and f_L is a 2-to-1 branched cover onto

a line.
iii) C is a generic conic iff f_C is generically 1-to-1 onto a curve of degree 4 with three singular points each having inverse image of cardinality 2.
iv) C is a jumping conic of jump size 1 iff f_C is 1-to-1 onto a curve of degree 4 which is smooth except for a lone singular point τ for which $f_C^{-1}(\tau)$ consists of 3 points.
v) C is a jumping conic of jump size 2 iff $C = Z_{\sigma\wedge s}$ for σ and s global sections of $E(1)$ and, in this case, f_C is a 4-to-1 branched cover onto a line.

Now consider a bundle E of non-general type meaning that there is a line $L \subset \mathbb{P}^2$ and a $\sigma \in H^0(\mathbb{P}^2; E(1))$ with at least 3 zeros on L.

Proposition 8. *Let E be a rank 2 stable bundle on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 3$ of non-general type. Then there is a unique line L such that every global section of $E(1)$ has exactly 3 zeros on L. E can be obtained from $T\mathbb{P}^2$ by an elementary modification of the form*

$$0 \longrightarrow E(1) \longrightarrow T\mathbb{P}^2 \longrightarrow j_{L*}\mathcal{O}_L(4) \longrightarrow 0. \tag{6.5}$$

Proof. There is a line L and $\sigma \in H^0(\mathbb{P}^2; E(1))$ such that σ has 3 zeros on L. All 4 of the zeros of σ can not be on L or else the cohomology sequence of

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\sigma} E \xrightarrow{\sigma\wedge} \mathcal{I}_{Z_\sigma}(1) \longrightarrow 0$$

would imply $h^0(\mathbb{P}^2; E) = h^0(\mathbb{P}^2; \mathcal{I}_{Z_\sigma}(1)) = 1$ contradicting stability. Clearly $E(1)_L \cong \mathcal{O}_L(3) \oplus \mathcal{O}_L(-1)$ and therefore every global section of $E(1)$ has 3 zeros on L. Perform the elementary modification

$$0 \longrightarrow E' \longrightarrow E(1) \xrightarrow{\phi} j_{L*}\mathcal{O}_L(-1) \longrightarrow 0 \tag{6.6}$$

where $\phi(s) \equiv \mathcal{O}_L(-1)$ – component of $s\,|_L$. By (2.4) we have $c_1(E') = 1$ and $c_2(E') = 1$ and therefore $c_1(E'(1)) = 3$ and $c_2(E'(1)) = 3$. Since E' is also stable, it follows that $E'(1) \cong T\mathbb{P}^2$ [10, Sec.8]. By the "inverse" elementary modification [4, page 41](or by taking the dual of (6.6)) one gets

$$0 \longrightarrow E \longrightarrow E' \xrightarrow{\psi} j_{L*}\mathcal{O}_L(3) \longrightarrow 0$$

and therefore

$$0 \longrightarrow E(1) \longrightarrow T\mathbb{P}^2 \xrightarrow{\psi} j_{L*}\mathcal{O}_L(4) \longrightarrow 0.$$

Since $T\mathbb{P}^2$ is globally generated, it follows that $E(1)$ is generated by global sections on $\mathbb{P}^2 \setminus L$. This proves the uniqueness of L. □

Note that ψ is given by

$$T\mathbb{P}^2 \to T\mathbb{P}^2|_L \cong \mathcal{O}_L(2) \oplus \mathcal{O}_L(1) \overset{\psi_2 \oplus \psi_3}{\longrightarrow} \mathcal{O}_L(4) \tag{6.7}$$

and so, counting parameters, $dim\{E\} = dim\{L\} + dim\{(\psi_2, \psi_3)\} - \dim\, Aut(\mathcal{O}_L(4)) = 2 + 3 + 4 - 1 = 8$ (Recall that dim $\mathcal{M}(0, 3) = 9$.).

We describe the jumping lines and conics in terms of the elementary modification (6.5). Clearly L is a jumping line of jump size 2. For any other line L_1, set $p = L \cap L_1$ and restrict (6.5) to L_1 to get

$$0 \longrightarrow E(1)_{L_1} \longrightarrow T\mathbb{P}^2_{L_1} \xrightarrow{\psi_{L_1}} \mathbf{C}p \longrightarrow 0$$

where $\mathbf{C}p$ is $\mathbf{C}$ at p and 0 elsewhere. Examine ψ_{L_1} by choosing a splitting of $T\mathbb{P}^2_{L_1}$

$$T\mathbb{P}^2_{L_1} \cong TL_1 \oplus NL_1 \cong \mathcal{O}_{L_1}(2) \oplus \mathcal{O}_{L_1}(1) \to \mathcal{O}_{L_1}(2)_p \oplus \mathcal{O}_{L_1}(1)_p \to \mathbf{C}. \tag{6.8}$$

Set $K_p = \operatorname{Ker} \psi_{L_1}$. From the definition of the elementary modification (6.5), one sees that $K_p = TL_{1p}$ iff $E(1)_{L_1} \cong \mathcal{O}_{L_1}(2) \oplus \mathcal{O}_{L_1}$ and $K_p \neq TL_{1p}$ iff $E(1)_{L_1} \cong \mathcal{O}_{L_1}(1) \oplus \mathcal{O}_{L_1}(1)$.

Recall that the jumping line locus $J_1 \subset \mathbb{P}^2$ is a curve of degree 3. The lines in $\mathbb{P}^{2^*}$ have the form $\Lambda_p \equiv$ lines in $\mathbb{P}^2$ through p and therefore, if $\Lambda_p \not\subseteq J_1$, $J_1 \cdot \Lambda_p = 3$, i.e. there are three jumping lines through each point p. If $p \in L$, the fact that the jump size of L is 2 implies that $(J_1 \cdot \Lambda_p)_l \geq 2$ for l the point in $\mathbb{P}^{2^*}$ corresponding to L. View ψ_p via a splitting of $T\mathbb{P}^2_L$ as in (6.7) and let $z, w \in L$ be the zeros of ψ_2. For $p \neq w, z$, $\operatorname{Ker} \psi_p \neq TL_p$ and so, by the discussion of the previous paragraph, there is a jumping line through p transverse to L. For $p = z$ or w, $(J_1 \cdot \Lambda_p)_l = 3$. This shows that, for $z \neq w$, l is a node of J_1 and that Λ_z and Λ_w are the tangents to J_1 at l. If $z = w$, J_1 has a cusp at l.

For $p \in \mathbb{P}^2 \setminus L$ the jumping lines through p are determined as follows. For $a \neq b \in L$, parameterize L by $\gamma(t) = t_0 a + t_1 b$. The line $\overline{p\gamma(t)}$ is jumping iff $p \bmod \mathbf{C}\gamma(t) \in T\mathbb{P}^2_{\gamma(t)}$ spans $K_{\gamma(t)}$. Now $\xi(t) = p \bmod \mathbf{C}\gamma(t)$ defines a global section of $T\mathbb{P}^2_L(-1)$, in fact, it gives a never vanishing section of $NL(-1) \cong \mathcal{O}_{\mathbb{P}^1}$ and so $\psi_{\gamma(t)}(\xi(t)) = \psi_3(\gamma(t)) \in H^0(L; \mathcal{O}_L(3))$ is not identically zero. Its three zeros determine the jumping lines through p. We see that l is the only singular point of J_1 as follows. For $L_1 \neq L$ a jumping line and l_1 the corresponding point of $\mathbb{P}^{2^*}$, let $p = L \cap L_1$ as above. Then $J_1 \cdot \Lambda_p = 2l + l_1$ so $(J_1 \cdot \Lambda_p)_{l_1} = 1$. If l_1 were a singular point of J_1, $(J_1 \cdot \Lambda)_{l_1} \geq 2$ for all lines through l_1. We have proven

Proposition 9. *Let E be a rank 2 stable bundle with $c_1 = 0$, $c_2 = 3$ of non-general type. The jumping lines form a cubic curve in $\mathbb{P}^{2^*}$ with exactly one singular (double) point corresponding to the line L (in Proposition 8).*

Let C be a smooth conic and as usual give it a quadratic parameterization $\mathbb{P}^1 \xrightarrow{\gamma} C$. The splittings

$$T\mathbb{P}^2_C \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \cong U \otimes \mathcal{O}_{\mathbb{P}^1}(3)$$

for $\dim U = 2$ follow from examination of the global sections of $T\mathbb{P}^2$. We determine if C is a jumping conic by noting the effect of the elementary modification (6.5) on this splitting. Let p and q be the intersection points of C and L and assume they are distinct. Restricting (6.5) to C, we get

$$0 \longrightarrow E(1)_C \longrightarrow T\mathbb{P}^2_C \xrightarrow{\tilde{\psi}_p \oplus \tilde{\psi}_q} \mathbf{C}p \oplus \mathbf{C}q \longrightarrow 0$$

where $\tilde{\psi}_p$ is obtained from the homomorphism ψ of (6.7) evaluated at p and similarly for $\tilde{\psi}_q$. Therefore $K_p \equiv \operatorname{Ker} \tilde{\psi}_p$ and $K_q \equiv \operatorname{Ker} \tilde{\psi}_q$ can be considered as 1-dimensional subspaces of U. If $K_p = K_q$, $E(1)_C$ contains a copy of $\mathcal{O}_{\mathbb{P}^1}(3)$ and so $E(1)_C \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. If $K_p \neq K_q$, $E(1)_C$ contains a copy of $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ and so has this splitting.

Now suppose C is tangent to L at p. By a change in the homogeneous coordinates of $\mathbb{P}^2$ we can assume that $p = e_0$ ($p = (0,0)$ in the corresponding affine coordinates x_1, x_2) and L is defined by $x_2 = 0$. By changing coordinates on a smaller Zariski open neighborhood of p we can also assume that C is defined by $x_2 + x_1^2 = 0$ and therefore x_1 is a local coordinate of C near p. Restricting (6.5) to C we have an elementary modification at the divisor $2e_0$

$$0 \longrightarrow E(1)_C \longrightarrow T\mathbb{P}^2_C \xrightarrow{\Psi} \mathbf{C}[x_1]/(x_1^2) \longrightarrow 0$$

where $\Psi\colon T\mathbb{P}^2_C \cong U \otimes \mathcal{O}_{\mathbb{P}^1}(3) \to \mathbf{C}[x_1]/(x_1^2)$ is determined by the one-jet $j_1(\psi)$ of ψ at p (see (6.7)) as follows. With respect to a basis for U and the dual basis for U^* and for f a local section of $T\mathbb{P}^2_C$,

$$f(x_1) = \begin{pmatrix} f_1(x_1) \\ f_2(x_1) \end{pmatrix}, j_1(\psi) = \begin{pmatrix} a_{10} + a_{11}x_1 \\ a_{20} + a_{21}x_1 \end{pmatrix}$$

and

$$\Psi(f) = j_1((a_{10} + a_{11}x_1) f_1(x_1) + (a_{20} + a_{21}x_1) f_2(x_1)). \tag{6.9}$$

We show that C is a jumping conic if and only if the 1-jets $a_{10} + a_{11}x_1$ and $a_{20} + a_{21}x_1$ are scalar multiples iff $a_{10}a_{21} - a_{11}a_{20} = 0$ iff ker $\psi(0) =$ ker $\psi'(0)$ as subspaces of U. To verify this, change basis in U so that $a_{10} = 0$ and $a_{20} = 1$ and therefore the above condition is $a_{11} = 0$. Let t_0, t_1 be homogeneous coordinates of a quadratic parameterization $\mathbb{P}^1 \to C$ with $[1, 0]$ corresponding to p, $t \equiv t_1/t_0$ the affine coordinate on $\mathcal{U}_0 \equiv \mathbb{P}^1 \setminus \infty$, and $s \equiv 1/t$ the coordinate on $\mathcal{U}_1 \equiv \mathbb{P}^1 \setminus 0$. Then $f \in T\mathbb{P}^2$ is in $E(1)$ iff $\Psi(f) = 0$ iff

$$\begin{aligned} f_2(0) &= 0 \\ f_2'(0) + a_{11} f_1(0) &= 0. \end{aligned} \tag{6.10}$$

Assume ker $\psi(0) =$ ker $\psi'(0)$, that is, $a_{11} = 0$. Then

$$f(t) = \begin{pmatrix} f_1(t) \\ t^2 h_2(t) \end{pmatrix}$$

and this implies $E(1)_C \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Assuming instead that $a_{11} \neq 0$,

$$f(t) = \begin{pmatrix} -a_{11}^{-1} f_{21} + f_{11} t + t^2 h_1(t) \\ f_{21} t + t^2 h_2(t) \end{pmatrix} = \begin{pmatrix} t & -a_{11}^{-1} \\ 0 & t \end{pmatrix} \cdot \begin{pmatrix} k_1(t) \\ k_2(t) \end{pmatrix}$$

where $k_1(t) = f_{11} + t h_1(t) + a_{11}^{-1} h_2(t)$ and $k_2(t) = f_{21} + t h_2(t)$. The transition function for $E(1)$ (from the $\mathcal{U}_0$ to the $\mathcal{U}_1$ trivialization) is therefore

$$\begin{pmatrix} t^{-3} & 0 \\ 0 & t^{-3} \end{pmatrix} \cdot \begin{pmatrix} t & -a_{11}^{-1} \\ 0 & t \end{pmatrix} = \begin{pmatrix} t^{-2} & -a_{11}^{-1} t^{-3} \\ 0 & t^{-2} \end{pmatrix} = \begin{pmatrix} 1 & -a_{11}^{-1} s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t^{-2} & 0 \\ 0 & t^{-2} \end{pmatrix}.$$

Changing the trivialization of $E(1)$ over $\mathcal{U}_1$ via the inverse of the matrix function of s above, we see that $E(1)_C \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$.

7. Rank 2 Stable Bundles on $\mathbb{P}^2$ with $c_1 = -1$ and $c_2 = 2$

The analysis of jumping lines and conics for a rank 2 stable bundle E on $\mathbb{P}^2$ with $c_1 = -1$ and $c_2 = 2$ is very similar to the study of the $c_1 = 0$, $c_2 = 3$ non-general type case. We summarize the results. E has a unique jumping line L characterized by: $L = Z_{\sigma\wedge s}$ for σ, s a basis for $H^0(\mathbb{P}^2; E(1))$. E can be expressed as an elementary modification along L of the trivial bundle

$$0 \longrightarrow E \longrightarrow U \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\psi} j_{L*}\mathcal{O}_L(2) \longrightarrow 0 \tag{7.1}$$

for U a vector space of dimension 2. A smooth conic C which intersects L at distinct points p and q is a jumping conic iff ker $\psi_p =$ ker ψ_q as subspaces of U. If C and L have a double intersection at p, C is a jumping conic iff ker $\psi_p =$ ker ψ'_p. Furthermore, if homogeneous coordinates are chosen so that L is defined by $x_2 = 0$ and if ψ is given by $\psi_k = \psi_{k00}x_0^2 + \psi_{k01}x_0x_1 + \psi_{k11}x_1^2$ for $k = 1, 2$, then the hyperplane $J_2 \subset \mathbb{P}^5$ is given by

$$a_{00}\xi_{00} + a_{01}\xi_{01} + a_{11}\xi_{11} = 0$$

where C is defined by $\sum_{0\le i\le j\le 2} \xi_{ij}x_ix_j = 0$ and $a_{00} = \psi_{101}\psi_{211} - \psi_{111}\psi_{201}$, $a_{01} = \psi_{111}\psi_{200} - \psi_{100}\psi_{211}$, and $a_{11} = \psi_{100}\psi_{201} - \psi_{101}\psi_{200}$.

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